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# A CASE OF WELL-DEFINED THERMAL DERIVATIVE EXPANSION TO LOWEST ORDER

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**Abstract**

We examine a very simple model for which the leading contribution to the one-loop effective potential at finite temperature is uniquely defined despite the presence of the Landau terms. In addition we report on the usual non-analyticity at finite temperature in order to compare our perturbative results with exact ones obtained in the literature. Finally, we point out the significance of our conclusions in the context of symmetry restoration at finite temperature.

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# 1 Introduction

It is well-known that for most of the theories and models the effective action displays a non-analytic behaviour at finite temperature [1, 2] and this puts in jeopardy the construction of an effective potential based on the derivative expansion technique [3, 4]. Historically this problem was first pointed out in the BCS theory context by Abrahams and Tsuneto [5] and later was also seen to appear in the gluon [6] and in the photon self-energy [7] for example.

The reason for this behavior is that temperature effects give rise to Landau terms and these are responsible for the development of a new branch cut in the complex plane of the external momenta with a branch point at the origin, besides the one already present at zero temperature. Then we have two branch cuts, the usual one defined by

$$ {q_0}^2 - |\mathbf{q}|^2 \geq 4m^2 $$

and another which present only at finite temperature, namely

$$ {q_0}^2 - |\mathbf{q}|^2 \leq 0 $$

.

This second branch cut is not common to all graphs at finite temperature. In fact it was shown that whenever the internal propagators in a typical loop have different masses the branch point is not in the origin, and the branch cut extends from $(m_1 - m_2)^2$ to $-\infty$ [8] where $m_1$ and $m_2$ are the masses of the particles in the internal loop.

In this letter we present another model where at least one of the graphs displays analytic behaviour at zero external momenta. We show in two different ways that the limit is well-defined and explain why this is so. We also show how the non-analyticity can develop in another model and compare our results with those obtained using non-perturbative techniques.

In section 2, we present the model and examine carefully the behaviour of the boson self-energy bubble diagram as the external momenta go to zero. In section 3, we replace the parity conserving interaction term of the theory with a similar but parity violating one and examine the consequences.

## 2 A new case

We consider the following model

$$L[\bar{\psi},\psi,\phi] = \bar{\psi}(i\not\partial - m)\psi - ig\bar{\psi}\gamma_5\psi\phi + L_0[\phi] \tag{1}$$

where $L_0[\phi]$ is the free Klein-Gordon Lagrangian. The boson is taken to be a pseudo-scalar quantity.

We consider $\phi(x)$ to be an external field and we want to obtain the one loop contribution to the effective action which is given by

$$\Gamma_{eff}[\phi] = -i\ln\frac{\mathrm{Det}[iS^{-1}[\phi]]}{\mathrm{Det}[iS^{-1}]} \tag{2}$$

where $iS^{-1}[\phi]$ and $iS^{-1}$ are matrices whose elements in coordinate representation are

$$\begin{aligned}\langle x|iS^{-1}|y\rangle &= (i\not\partial_x - m)\delta(x-y)\\ \langle x|iS^{-1}[\phi]|y\rangle &= (i\not\partial_x - m - ig\gamma_5\phi(x))\delta(x-y)\end{aligned}$$

Since the external field depends on the coordinates, the resulting functional determinants are not straightforward to calculate. The matrices whose functional determinants we want to evaluate are not diagonal in momentum or in coordinate space. However, we can write equation (2) as

$$\Gamma_{eff}[\phi] = -i\mathrm{Tr}\ln\left[1 - ig\gamma_5(-iS)\phi(x)\right]. \tag{3}$$

Now we expand it in powers of the coupling constant and show that the leading contribution to the one-loop effective action is

$$\Gamma^{(2)} = \frac{ig^2}{2}\int\frac{d^4q}{(2\pi)^4}\tilde{\phi}(-q)i\Pi(q)\tilde{\phi}(q),$$

where

$$i\Pi(q) = -\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left[\gamma_5\frac{1}{\not k + \not q - m}\gamma_5\frac{1}{\not k - m}\right] \tag{4}$$

We note that $i\Pi(q)$ is just the self-energy bubble diagram for the boson which, after performing the trace, is given by

$$i\Pi(q) = 4\int\frac{d^4k}{(2\pi)^4}\frac{k^2 + k^\mu q_\mu - m^2}{[(k+q)^2 - m^2][k^2 - m^2]}. \tag{5}$$

This is one typical diagram that usually has a non-analytical behavior in the limit of vanishing external momenta, but we are going to show that this is not the case here. We keep this intermediate expression because it will help us to show in the next section how the non-analyticity can develop in the scalar-coupling model.

It is worth mentioning that the leading contribution to the one-loop effective action can also be written as

$$\Gamma^{(2)} = \frac{ig^2}{2} \int d^4x \; \phi(x) i\Pi(q)\phi(x),$$

where $q_\mu$ is to be understood as a derivative operator acting on the field to the right. This result can be obtained by inserting complete sets of eigenvectors of momentum and position operators. The external field is an operator acting on coordinate states which at finite temperature are thermal states. Then the eigenfunction $\phi(x)$ has to be interpreted as being thermalized itself. This seems to us to be the reason why the approach given in [9, 10] is not in agreement with what is usually expected for the effective action at finite temperature.

Applying the usual finite temperature techniques to (5), we find the following expression for the thermal bubble diagram.

$$\begin{aligned}\Pi(q_0,\mathbf{q}) = &-\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega\Omega}\Bigg\{4\omega\tanh\frac{\beta\Omega}{2}+ \\ &+\frac{1}{\Omega+\omega-q_0}\left[[\omega q_0+\mathbf{kq}]\tanh\frac{\beta\omega}{2}+[q_0^2-\Omega q_0+\mathbf{kq}]\tanh\frac{\beta\Omega}{2}\right] \\ &+\frac{1}{\Omega+\omega+q_0}\left[[-\omega q_0+\mathbf{kq}]\tanh\frac{\beta\omega}{2}+[q_0^2+\Omega q_0+\mathbf{kq}]\tanh\frac{\beta\Omega}{2}\right] \\ &+\frac{1}{\Omega-\omega+q_0}\left[[\omega q_0+\mathbf{kq}]\tanh\frac{\beta\omega}{2}-[q_0^2+\Omega q_0+\mathbf{kq}]\tanh\frac{\beta\Omega}{2}\right] \\ &+\frac{1}{\Omega-\omega-q_0}\left[[-\omega q_0+\mathbf{kq}]\tanh\frac{\beta\omega}{2}-[q_0^2-\Omega q_0+\mathbf{kq}]\tanh\frac{\beta\Omega}{2}\right]\Bigg\},\end{aligned} \tag{6}$$

where

$$\omega = \sqrt{\mathbf{k}^2+m^2} \qquad\qquad \Omega = \sqrt{(\mathbf{k}+\mathbf{q})^2+m^2}\ .$$

We can have a first indication that the zero-momentum limit of expression (6) does not display the usual non-uniqueness problem by examining the two successive limits. Namely,

$$\begin{aligned}\Pi(0,\mathbf{q}) &= -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\omega\Omega}\left\{2\omega\tanh\frac{\beta\Omega}{2}+\frac{\mathbf{kq}}{\Omega+\omega}\left[\tanh\frac{\beta\omega}{2}+\tanh\frac{\beta\Omega}{2}\right]\right.\\ &+ \left.\frac{\mathbf{kq}}{\Omega-\omega}\left[\tanh\frac{\beta\omega}{2}-\tanh\frac{\beta\Omega}{2}\right]\right\},\end{aligned}$$

which can be checked to give

$$\lim_{|\mathbf{q}|\to 0}\Pi(0,\mathbf{q}) = -\frac{1}{\pi^2}\int_{|m|}^{\infty} d\omega\sqrt{\omega^2-m^2}\tanh\frac{\beta\omega}{2}\ . \tag{7}$$

Similarly we find

$$\begin{aligned}\Pi(q_0,0) &= -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega\Omega}\left\{4\omega\tanh\frac{\beta\Omega}{2}+\right.\\ &+ \left.{q_0}^2\left[\frac{1}{2\omega-q_0}+\frac{1}{2\omega+q_0}\right]\tanh\frac{\beta\omega}{2}\right\}.\\ &\xrightarrow{q_0\to 0} -\frac{1}{\pi^2}\int_{|m|}^{\infty} d\omega\sqrt{\omega^2-m^2}\tanh\frac{\beta\omega}{2}\ .\end{aligned}$$

We conclude that the limits coincide. Moreover, the only term that contributes to the unique result is the first one in the integrand of equation (6) and those proportional to Landau terms - the last two terms inside the integrand - vanish in this limit.

A more general way of seeing that the limits are the same is to perform the parameterization $q_0 = a|\mathbf{q}|$, where $a$ can be any real number, and find the limit of $\Pi(a|\mathbf{q}|,|\mathbf{q}|)$ as $|\mathbf{q}|\to 0$. If the limit is independent of $a$, we have a strong indication that the function is analytic at the origin, *i.e.* it does not depend on the way one approaches the origin [1]. Before doing so we recast equation (6) in a more convenient form by means of the transformation $\mathbf{k}\to -(\mathbf{k}+\mathbf{q})$ wherever the integrand contains $\tanh\frac{\beta\Omega}{2}$. Then we find

$$\Pi = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\frac{2}{\omega}\tanh\frac{\beta\omega}{2}+(q_0^2-\mathbf{q}^2)\tanh\frac{\beta\omega}{2}\right.$$

$$\times \quad \frac{1}{2\omega\Omega}\left[\frac{1}{q_0+\Omega+\omega}-\frac{1}{q_0-\Omega-\omega}+\frac{1}{q_0+\Omega-\omega}-\frac{1}{q_0-\Omega+\omega}\right]\Bigg\} \tag{8}$$

One can note that at $T=0$ the Landau terms cancel each other, as expected. We change variables from $\cos\theta$ to $\Omega$ and perform the integration over $\Omega$. The result is

$$\begin{aligned}\Pi(a|\mathbf{q}|,|\mathbf{q}|) &= -\frac{1}{\pi^2}\int_{|m|}^{\infty} d\omega\sqrt{\omega^2-m^2}\tanh\frac{\beta\omega}{2} - \\ &- \frac{(a^2-1)|\mathbf{q}|^2}{2|\mathbf{q}|}\int_{|m|}^{\infty}\frac{d\omega}{(2\pi)^2}\tanh\frac{\beta\omega}{2}[L1+L2+L3+L4] \quad (9)\end{aligned}$$

where

$$L1(|\mathbf{q}|)=\ln\frac{\Omega_+ +\omega+a|\mathbf{q}|}{\Omega_- +\omega+a|\mathbf{q}|} \qquad L2(|\mathbf{q}|)=\ln\frac{\Omega_+ +\omega-a|\mathbf{q}|}{\Omega_- +\omega-a|\mathbf{q}|}$$
$$L3(|\mathbf{q}|)=\ln\frac{\Omega_+ -\omega+a|\mathbf{q}|}{\Omega_- -\omega+a|\mathbf{q}|} \qquad L4(|\mathbf{q}|)=\ln\frac{\Omega_+ -\omega-a|\mathbf{q}|}{\Omega_- -\omega-a|\mathbf{q}|}$$

with

$$\Omega_+ = \sqrt{(|\mathbf{k}|+|\mathbf{q}|)^2+m^2} \qquad \Omega_- = \sqrt{(|\mathbf{k}|-|\mathbf{q}|)^2+m^2}.$$

The limits of two of the regular terms $L1$ and $L2$ are independent of $a$ as they should be. We can see that

$$\lim_{|\mathbf{q}|\to 0}(a^2-1)|\mathbf{q}|L1=0 \qquad \lim_{|\mathbf{q}|\to 0}(a^2-1)|\mathbf{q}|L2=0.$$

What is quite unexpected is that, for this particular model, the contributions coming from the Landau terms, $L3$ and $L4$, vanish independently of $a$, that is

$$\lim_{|\mathbf{q}|\to 0}(a^2-1)|\mathbf{q}|L3=0 \qquad \lim_{|\mathbf{q}|\to 0}(a^2-1)|\mathbf{q}|L4=0.$$

In other words, although the Landau terms are not well-behaved at the origin of momentum space a unique effective potential can be defined thanks to the kinetic term in the numerator of equation (8), namely $q_0^2-\mathbf{q}^2$. This is an interesting result, but this kinetic term does not always appear in bubble

diagrams as we are going to see in the next section. In the present case the one-loop, $g^2$ order contribution to the effective potential is

$$\begin{aligned} V_{eff}^{(2)} &= -\frac{ig^2}{2} i\Pi(0,0)\phi^2 \\ \Pi(0,0) &= -\frac{1}{\pi^2}\int_{|m|}^{\infty} d\omega \sqrt{\omega^2 - m^2} \tanh\frac{\beta\omega}{2} \, . \end{aligned} \tag{10}$$

The next order in the derivative expansion is non-analytic since the derivatives of the Landau terms become dominant and the derivative expansion breaks down.

# 3 A Usual Case

If we replace in equation (1) the interaction term with one which does not contain the $\gamma_5$ matrix and repeat the same procedure we find

$$i\Pi^{'}(q) = 4\int \frac{d^4k}{(2\pi)^4} \frac{k^2 + k^{\mu}q_{\mu} + m^2}{[(k+q)^2 - m^2][k^2 - m^2]} \tag{11}$$

which can be written as

$$i\Pi^{'}(q) = i\Pi(q) + i\Pi^{''}(q),$$

where

$$i\Pi^{''}(q) = 4\int \frac{d^4k}{(2\pi)^4} \frac{2m^2}{[(k+q)^2 - m^2][k^2 - m^2]} \, .$$

As we saw in the previous section $\Pi(a|\mathbf{q}|, \mathbf{q})$ does not depend on $a$, when $\mathbf{q} \to 0$. On the other hand $\Pi^{''}(q)$ does. In fact, we have

$$\begin{aligned} \lim_{|\mathbf{q}|\to 0} \Pi^{''}(a|\mathbf{q}|, |\mathbf{q}|) = \frac{m^2}{\pi^2}\int_{|m|}^{\infty} d\omega \left\{ \frac{\sqrt{\omega^2 - m^2}}{\omega^2} \tanh\frac{\beta\omega}{2} - \right. \\ \left. - \frac{\beta}{2\omega}\cosh^{-2}\frac{\beta\omega}{2}\left[\sqrt{\omega^2 - m^2} - \frac{\omega a}{2}\ln\frac{|\omega a + \sqrt{\omega^2 - m^2}|}{|\omega a - \sqrt{\omega^2 - m^2}|}\right]\right\} . \end{aligned} \tag{12}$$

We see that dropping $\gamma_5$ from the interaction has made a great difference which is reflected on the relative signs of the terms in the numerator (Compare equation (5) to equation (11).).

It is important to compare our results with other in the literature and, in particular, with Dolan and Jackiw [11] . For the one-loop effective potential at order $g^2$ we have

$$V_{eff}^{(2)} = -\frac{g^2}{2\pi^2}\int_{|m|}^{\infty} d\omega \left\{ \sqrt{\omega^2 - m^2}\left(1 - \frac{m^2}{\omega^2}\right)\tanh\frac{\beta\omega}{2} + \frac{\beta}{2\omega}\cosh^{-2}\frac{\beta\omega}{2}\left[\sqrt{\omega^2 - m^2} - \frac{\omega a}{2}\ln\frac{|\omega a + \sqrt{\omega^2 - m^2}|}{|\omega a - \sqrt{\omega^2 - m^2}|}\right]\right\} \phi^2(x). \quad (13)$$

This gives the contribution for the thermal mass of the $\phi(x)$ field. Considering the external field to be constant Dolan and Jackiw obtained the following exact expression for the one-loop effective potential

$$V_{eff} = -\frac{2}{\pi^2}\int_{|m|}^{\infty} d\omega\sqrt{\omega^2 - m^2}\left[\frac{E}{2} + \frac{1}{\beta}\ln\left(1 - e^{\beta E}\right)\right], \quad (14)$$

where

$$E = \left[\sqrt{\omega^2 - m^2} + (m + g\phi)^2\right]^{1/2}.$$

We are interested in the contribution at the second order in the coupling constant which is

$$\begin{aligned} V_{eff}^{(2)} &= -\frac{g^2}{2\pi^2}\int_{|m|}^{\infty} d\omega\sqrt{\omega^2 - m^2}\left\{\left(1 - \frac{m^2}{\omega^2}\right)\tanh\frac{\beta\omega}{2} + \right. \\ &+ \left. \frac{m^2\beta}{2\omega}\cosh^{-2}\frac{\beta\omega}{2}\right\} \phi^2. \end{aligned} \quad (15)$$

If we set $a = 0$ in equation (13) it reduces to expression (15), which means that the result derived by Dolan and Jackiw is valid only in one of the infinite number of ways of approaching the origin, namely in the case where we take $q_0 \to 0$ first and then $\mathbf{q} \to 0$. One can also reproduce equation (15) by setting $(q_0, \mathbf{q}) = (0, 0)$ in formula (11) and then performing the Matsubara sum.

This is also equivalent to assuming from the beginning that the derivative expansion is well defined at finite temperature. However, the correct thing to do is to perform the sum first and then see how $\Pi(q)$ behaves in the limit $(q_0, \mathbf{q}) \to (0, 0)$. We therefore conclude that the non-perturbative method employed in [11] is not generally equivalent to the perturbative calculation because it fails to take into account the non-analyticity which appears at the origin of the space of external momenta.

# 4 Conclusions

We have shown that in a model, where fermions couples to a pseudo-scalar field, the thermal mass for the pseudo-scalar field can be found uniquely at finite temperature. We have also shown that this is not true when the fermion couples to a scalar field, the reason for that being the non-analytic behaviour which appears in the thermal bubble diagram. The models we dealt with can be considered together to study chiral symmetry restoration at finite temperature for example in the linear $\sigma$ model [12] in its broken chiral symmetry phase and in the Nambu-Jona-Lasinio model [13] expressed in terms of auxiliary fields.

Finally, we point out that, whenever finite temperature chiral symmetry restoration is discussed by employing non-perturbative results for the effective potential, they may not match those based on perturbation theory. Therefore the question of symmetry restoration at finite temperature should be reanalyzed keeping in mind the non-analyticity of some graphs. Work on this and other related issues is in progress.

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